A REMARK ON ALMOST PERIODIC TRANSITION OPERATORS

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ABSTRACT

Results of Rosenblatt on almost periodic transition operators are extended to the reducible case.

Let X be a compact Hausdorff space and T a non-negative linear operator on $C(X)$ with $T1 = 1$. Such an operator defines (and is defined by) a weak* continuous map $x \to t_x$ from X into $P(X)$ (the space of probability measures on X) given by $Tf(x) = t_x(f)(= \int f(y)t_x(dy))$, $f \in C(X)$. We shall call the closure of the union of the supports of the measures t_x the support of T, denoted by Σ_T.

Recently Rosenblatt [4, 5] has considered the (admittedly rare) situation in which $S = \{T^n: n \geqq 1\}$ is almost periodic in the sense that the orbit $\{T^n f: n \geqq 1\}$ is conditionally compact in $C(X)$ for *all* f in $C(X)$. From [2] one knows that then $C(X) = C_0 \oplus C_p$, where C_0 is the closed invariant subspace consisting of all f with $\| T^n f \| \to 0$, and C_p is the closed invariant span of the eigenvectors of T with eigenvalues of unit modulus. In [5] Rosenblatt showed that, when T is irreducible in a certain sense, C_p can be identified with $C(Y)$ for some quotient space Y of X, while T is induced on $C(Y)$ by a self-homeomorphism of Y. We wish to point out a very simple derivation of a stronger assertion, which in Rosenblatt's context says Y is a compact monothetic group on which his self-homeomorphism is a translation.

To begin we recall that in the strong operator topology the closure $\bar{S}$ of $S = \{T^n: n \geqq 1\}$ is a compact abelian semigroup whose kernel (least ideal) K is a compact topological group [2]. Indeed it is precisely the identity E of K which projects $C(X)$ onto C_p and annihilates C_0. Naturally the elements of K are non-negative and leave 1 fixed, so setting $e_x(f) = Ef(x)$, $f \in C(X)$, defines $e_x \in P(X)$. Evidently $C_p = EC(X)$ is conjugate closed.

With Σ_E the support of E, $A = C_p | \Sigma_E = EC(X) | \Sigma_E$ and C_p are isometric, so A is closed in $C(\Sigma_E)$: for $|e_x(f)| \leqq \sup |f(\Sigma_E)|$, and, applied to Ef, $|e_x(f)| = |e_x(Ef)| \leqq \sup |Ef(\Sigma_E)|$, whence $\| Ef \| \leqq \| (Ef | \Sigma_E) \|$. As a consequence Rosenblatt's argument [4] shows A is a subalgebra of $C(\Sigma_E)$, viz.: A^R (the set of

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real elements of A) is a sublattice of $C^R(\Sigma_E)$ since for $E\phi_i = \phi_i \in A^R$, $i = 1,2$, $E(\phi_1 \vee \phi_2) \geqq E\phi_i = \phi_i$, $i = 1,2$, so $\psi = E(\phi_1 \vee \phi_2) - (\phi_1 \vee \phi_2) \geqq 0$; and since $E\psi = 0$, ψ vanishes on Σ_E. So by Stone's proof of the Stone-Weierstrass theorem, A^R (and thus A) is an algebra.

Consequently $A = C(Y)$ for some factor space Y of Σ_E. Now trivially $k = kE \in K$ has support $\Sigma_k \subset \Sigma_E$, since if $f \in C(X)$ vanishes on Σ_E then $kf = kEf = k0 = 0$. Since C_p is invariant for each k in K (as for all elemnts of $\bar{S}$), k actually yields a well defined operator $(f|\Sigma_E \to kf|\Sigma_E)$ on $A = C_p|\Sigma_R$. So K acts as a group of operators on A, and evidently $k \to kf$ is strongly continuous as a map into A since it coincides with $k \to kg|\Sigma_E$ for any extension $g \in C(X)$ of f, and $k \to kg$ is strongly continuous. Viewed as operators on $C(Y)$ then, K is a group of non-negative operators leaving 1 fixed whose identity is the identity operator. So each adjoint k^* maps $P(Y)$ onto itself, and thus maps extreme points onto exteme points: with μ_y the unit mass at y, $k^*\mu_y = \mu_{k(y)}$ for some unique $k(y)$ in Y. But

$$(k, y) \to k(y) \tag{1}$$

is continuous since this amounts to continuity of

$$(k, y) \to f(k(y)) = k^*\mu_y(f) = kf(y), \text{ all } f \in C(Y),$$

and that follows from the strong continuity of $k \to kf$. Hence each element k of K induces a self-homeomorphism $k(\cdot)$ of Y, which in turn induces the action of k on $C(Y)$: $kf = f \circ k(\cdot)$. Thus K, with the action (1), gives rise to a transformation group on Y which yields the action of K on $A = C_P|\Sigma_E$, and in particular that of $k_0 = TE$.

Having identified $C(Y)$ and $EC(X)|\Sigma_E$ we can of course compose $Ef|\Sigma_E$ with an element $k(\cdot)$ of our transformation group on Y, and thus write, without ambiguity,(1) $kf|\Sigma_E = k(f|\Sigma_E) = (f|\Sigma_E) \circ k(\cdot)$ if $f = Ef$. To obtain this action of T (hence that of $TE = k_0$) on $C_p = EC(X)$ then we note that for any x in X and $f = Ef$, $Tf(x) = TEf(x) = k_0 f(x) = Ek_0 f(x) = e_x(k_0 f|\Sigma_E)$, so

$$Tf(x) = e_x([f|\Sigma_E] \circ k_0(\cdot)), \qquad f \in EC(X). \tag{2}$$

Noting that the powers of $k_0 = TE$ are dense in $K = \bar{S}E$ since those of T are dense in $\bar{S}$ we have proved half of the following

THEOREM. *A non-negative operator T with $T1 = 1$ has $S = \{T^n : n \geqq 1\}$ almost periodic if and only if*

(i) *there is a projection E in the strong operator closure of S,*

(ii) *there is a quotient space Y of Σ_E for which $EC(X)|\Sigma_E$ is precisely $C(Y)$ (as naturally imbedded in $C(\Sigma_E)$),*

(iii) *a compact (monothetic) transformation group K acts on Y, with the action of T on $EC(X)$ that induced by a generator k_0 of K, as in (2).*

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(1) The second k is our operator on $A = C_p|\Sigma_E$.

"If" is easily proved by showing conditional compactness of orbits for f in $(I-E)C(X)$, and then for f in $EC(X)$, as follows.

Suppose the net $T^{n_\delta} \to E$ strongly. For $f \in (I-E)C(X)$, $Ef = 0$, so given $\varepsilon > 0$, $\| T^n f \| \leqq \| T^{n_\delta} f \| < \varepsilon$ for $n \geqq n_\delta$, some δ, whence $\| T^n f \| \to 0$ and our conditional compactness is apparent.

On the other hand, action of T on $EC(X)$ is determined on Σ_E, by (2), which also shows $EC(X)$ and $EC(X) \mid \Sigma_E$ are isometric. Thus our conditional compactness will follow from that of the corresponding orbit in $C(Y)$, which itself follows directly from the compactness of K and the fact that $k \to f \circ k(\cdot)$ is strongly continuous for any transformation group on a compact space.

Remarks. 1. In case $S = \{T^n : n \geqq 1\}$ is weakly almost periodic (i.e., $\{T^n f : n \geqq 1\}$ is conditionally weakly compact for each f in $C(X)$), (i) – (iii) still hold if "strong" is replaced by "weak" in i). Indeed E is then the identity of the least ideal K of the weak operator closure of S, which is still [2, 8.1] a compact group in the strong operator topology, so that the same proof applies. Needless to say, in this situation C_0 (the nullity of E) is not so simply described.

(More generally the same proof yields (i)–(iii)) (with obvious modifications) for any (weakly) almost periodic semigroup S of non-negative T with $T\,1 = 1$ for which the conclusions of [2, 4.11] hold (in particular for S amenable [1]), with C_0 and C_p the subspaces defined in [2].)

2. Note that if $X = \Sigma_E$, the natural decomposition of Y into orbits lifts to a decompositon $\mathscr{F}$ of X for which $x \in F \in \mathscr{F}$ implies the support of t_x is contained in F. Indeed if $f \in C(Y)$, $0 \leqq f \leqq 1$, and $f = 1$ on the orbit of the image y of x then viewing f as an element of C_p we have $Tf(x) = f(k_0(y)) = 1 = t_x(f)$, so that t_x is carried by $f^{-1}(1)$, hence by an arbitrary neighborhood of F if f is chosen appropriately, which yields the assertion.

Thus we may define operators $T_F : C(F) \to C(F)$ for which $Tf \mid F = T_F(f \mid F)$, $f \in C(X)$, and so in a sense decompose T into irreducible parts. As is easily seen the case in which $\mathscr{F}$ is a singleton is precisely Rosenblatt's irreducible case, and then Y and K can be identified.

3. That identification is made possible by the fact that our operator group K, in its role as a transformation group on Y, always acts effectively; i.e., $k(y) = y$ for all y implies $k = E$. For it clearly implies $kf = f$ for all f in C_p, whence $kf = kEf = Ef$ for all f in $C(X)$.

4. Whenever C_p is finite dimensional, as in the special case in which T is a compact operator (where S is necessarily almost periodic), one easily obtains parallels to the results obtainable when T is compact [3, §8]. Indeed if $f_1, \cdots, f_n$ are independent eigenfunctions spanning C_p, corresponding to eigenvalues $\lambda_1, \cdots, \lambda_n$, the fact that K must be finite (since Y is, and K is effective) shows TE is of finite order $N \leqq n!$ and each λ_i is a root of unity. Moreover since Ef is uniquely expres-

sible as $\sum_1^n c_i(f)f_i$ while $f \to c_i(f)$ is continuous, we have unique complex measures μ_i with

$$(3) \qquad Ef = \sum_{i=1}^{n} \mu_i(f)f_i, \qquad f \in C(X),$$

necessarily biorthogonal to the f_i. And $T^*\mu_i = \lambda_i\mu_i$: for given any f in $C(X)$, $g = \Sigma\lambda_j^{-1}T^*\mu_j(f)f_j - Ef$ is an element of C_p with $Tg = \Sigma T^*\mu_j(f)f_j - TEf$ $= \Sigma\mu_j(Tf)f_j - TEf = ETf - TEf = 0$ since $\bar{S}$ is commutative. But T acts as an invertible on C_p, so $g = 0$, $Ef = \Sigma\lambda_j^{-1}T^*\mu_j(f)f_j$, whence $T^*\mu_j = \lambda_j\mu_j$ by uniqueness of the μ_j.

Lastly, from the fact that $T^NE = (TE)^N = E$ one concludes that $T^{Nj} \to E$ strongly as $j \to \infty$; for any strong cluster point k of the sequence must lie in K, as is easily seen, so that $T^{Nj}E = E$ implies $k = kE = E$. Since $\bar{S}$ is compact in the strong operator topology, our convergence is assured.

5. Finally, we note that invariant integration over K can be used to obtain analogues of the other results of Rosenblatt [5], while the eigenfunctions f spanning $C_p|\Sigma_E$ are easily obtained; each is a (common) eigenfunction (with unimodular eigenvalue) of each k in K, and since $k \to kf$ is continuous, $kf = \chi(k)f$ for some character χ of K. Thus each such f coincides on each orbit in Y with a multiple of a fixed character χ_f of K, and any such function is obviously an eigenfunction in $C_p|\Sigma_E$. (Values off Σ_E must of course be computed via (2).)

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